Unidad de Investigación, Report

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1 Introduction

Unidad de Investigación, is the name of a course taught at the University of Chile. This course is based on the student researching a topic with the guidance of a professor, this academic is called a guiding professor, who is also in charge of evaluating the student. This report is from an *Unidad de Investigación*.

To begin with, in a Euclidean space, the symmetries of a convex polytope with composition form a finite group. From this, a natural question would be: ¿For every finite group G there exists a convex polytope M, in some Euclidean space, whose symmetry group is G? The answer to this question is affirmative; moreover, the purpose of this report is to detail a demonstration of Marcos Cossarini[2], that is, we walk through each step in more detail and we prove the two facts assumed in that proof.

The above question is answered by constructing for each finite group a convex polytope, so someone could use this to obtain faithful representation of groups on real vectors spaces of finite dimensions. That is, someone could visualize every finite groups within some Hilbert space, which is closer to reality than abstract groups.

Before starting, the reader should consider that this report uses the following knowledge: Linear Algebra, Metric spaces and group theory. Furthermore, topics related to affine spaces[1] and convex sets[4] are mentioned.

Now we start by reviewing everything needed for the Marcos Cossarini's proof.

2 Symmetries

In this section we specify the concept of $\mathbb{R}^n$-symmetry that we use. Also we see results on symmetries that are essential for our goal.

Definition 2.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an isometry with the usual metric. Let $A \subset \mathbb{R}^n$ be non-empty. Then f is a symmetry of A iff $f(A) = A$. We denote the set of all the symmetries of A by $Sym(A)$.*

The following Theorem is a generalization of Lemma [3, 12.13.], the proof is analogous.

Theorem 2.1. *Let f be a isometry of $\mathbb{R}^n$. If $f(0) = 0$, then f is a Hilbert space isomorphism.*

Proof. Since f is an isometry, we immediately obtain:

$$||x|| = ||x - 0|| = ||f(x) - f(0)|| = ||f(x) - 0|| = ||f(x)||.$$

For all $x \in \mathbb{R}^n$. Hence $\langle x, y\rangle = \langle f(x), f(y)\rangle$ for all $x, y \in \mathbb{R}^n$. With this is immediate that $\{f(e_i)\}_{i=1}^n$ is an orthonormal basis of $\mathbb{R}^n$.

Now let $x \in \mathbb{R}^n$, we quickly obtain:

$$f(x) = \sum_{i=1}^{n} \langle f(x),\, f(e_i)\rangle f(e_i) = \sum_{i=1}^{n} \langle x,\, e_i\rangle f(e_i).$$

Since $x = \sum_{i=1}^{n} \langle x,\, e_i\rangle e_i$ was any element of $\mathbb{R}^n$, we conclude that f is a linear mapping. Finally, f is a monomorphism between $\mathbb{R}$-vector spaces with same dimension and therefore f is a vector space isomorphism that preserves the inner product. □

Corollary 2.1. *If f is a isometry of $\mathbb{R}^n$, then f is affine and bijective.*

Proof. Since f is an isometry, then $g(x) := f(x) - f(0)$ is also an isometry. Furthermore, $g(0) = 0$, thus by the previous theorem we have that g is a bijective linear mapping. Then $f(x) = g(x) + f(0)$ for all x. Since g is a bijective linear mapping, then f is affine and bijective. □

Observation 2.1. *In Definition 2.1, since $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a isometry, by last corollary f is bijective, and therefore f is a symmetry of $\mathbb{R}^n$. That is, $Sym(A) \subset Sym(\mathbb{R}^n)$ for all non-empty set $A \subset \mathbb{R}^n$.*

Proposition 2.1. *Let $A = \{C_1, \cdots, C_m\} \subset \mathbb{R}^n$ be a finite set. If f is a symmetry of $\mathbb{R}^n$, then $f\left(\sum_{i=1}^{m} \lambda_i C_i\right) = \sum_{i=1}^{m} \lambda_i f(C_i)$, for all $\lambda_1, \cdots, \lambda_m \in \mathbb{R}^n$ such that $\sum_{i=1}^{m} \lambda_i = 1$.*

Proof. Let $z := \sum_{i=1}^{m} \lambda_i C_i$. Since f is a symmetry of $\mathbb{R}^n$, by the previous corollary we have $f = g + b$ with g a linear mapping and b a constant. Therefore:

$$f(z) = g(z) + b = g\left(\sum_{i=1}^{m} \lambda_i C_i\right) + b = \sum_{i=1}^{m} \lambda_i [g(C_i) + b] = \sum_{i=1}^{m} \lambda_i f(C_i)$$

Thanks to the fact that $\sum_{i=1}^{m} \lambda_i = 1$ and g is a linear mapping. □

Observation 2.2. *Let $A \in \mathbb{R}^n$ be a non-empty convex set and $f \in Sym(\mathbb{R}^n)$. We necessarily have that $f(A)$ is convex. Since if $z \in (f(x), f(y))$ for some $f(x), f(y) \in f(A)$, then $z = (1-\lambda)f(x) + \lambda f(y) = f((1-\lambda)x + \lambda y) \in f(A)$.*

Corollary 2.2. *Let $A = \{C_1, \cdots, C_m\} \subset \mathbb{R}^n$ be a finite set. If f is a symmetry of A, then $f(z) = z$, where $z = \sum_{i=1}^{m} \frac{1}{m} C_i$.*

Proof. By the previous proposition we obtain:

$$f\left(\sum_{i=1}^{m} \frac{1}{m} C_i\right) = \sum_{i=1}^{m} \frac{1}{m} f(C_i).$$

Since f is a symmetry of A, we finally have:

$$f(z) = f\left(\sum_{i=1}^{m} \frac{1}{m} C_i\right) = \sum_{i=1}^{m} \frac{1}{m} f(C_i) = \sum_{i=1}^{m} \frac{1}{m} C_i = z.$$

□

Lemma 2.1. *Let $\{v_1, \cdots, v_n\} \subset \mathbb{R}^n$ a linearly independent set and let $x, y \in \mathbb{R}^n$. If $||x|| = ||y||$ and $||x - v_k|| = ||y - v_k||$ for all k, then $x = y$.*

Proof. By inner product properties we have (1) and (2) for all k.

$$\begin{aligned} ||x - v_k||^2 &= ||x||^2 - 2\langle x, v_k\rangle + ||v_k||, \quad (1)\\ ||y - v_k||^2 &= ||y||^2 - 2\langle y, v_k\rangle + ||v_k||. \quad (2)\end{aligned}$$

Since $||x|| = ||y||$, $||x - v_k|| = ||y - v_k||$, (1) and (2) for all k, we obtain that $\langle x, v_k\rangle = \langle y, v_k\rangle$ for all k. Since $Span(\{v_1, \cdots, v_n\}) = \mathbb{R}^n$, we conclude that $x = y$.

□

Theorem 2.2. *Let $\{w_0, \cdots, w_n\} \subset \mathbb{R}^n$ an affinely independent set and let $x, y \in \mathbb{R}^n$. If $||x - w_k|| = ||y - w_k||$ for all k, then $x = y$.*

Proof. Consider the translation $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $s \mapsto s - w_0$. We have that $\{t(w_1), \cdots, t(w_n)\}$ is a linearly independent set. Furthermore, we obtain (3) and (4) for all $k \in \{1, \cdots, n\}$:

$$\begin{aligned} ||t(x)|| = ||t(x) - t(w_0)|| = ||x - w_0|| &= ||y - w_0|| = ||t(y) - t(w_0)|| = ||t(y)||, \quad (3)\\ ||t(x) - t(w_k)|| = ||x - w_k|| &= ||y - w_k|| = ||t(y) - t(w_k)||. \quad (4)\end{aligned}$$

Then, by the previous lemma, $t(x) = t(y)$ and therefore $x = y$. □

Lemma 2.2. *Let $X = \{0 = x_0, \cdots, x_n\} \subset \mathbb{R}^n$ an affinely independent set and let $Y = \{y_0, \cdots, y_n\} \subset \mathbb{R}^n$ a set. If $d(x_i, x_j) = d(y_i, y_j)$ for all i, j, then there exists an isometry $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the property $F(x_k) = y_k$ for all k.*

Proof. We define:

$$\begin{aligned} F : \mathbb{R}^n &\longrightarrow \mathbb{R}^n,\\ F\left(x_0 + \sum_{i=1}^{n} \lambda_i(x_i - x_0)\right) &= F(x_0) + \sum_{i=1}^{n} \lambda_i(F(x_i) - F(x_0)).\end{aligned}$$

Where $F(x_i) = y_i$ and $\lambda_i \in \mathbb{R}$ for all i. Since X is an affine basis of $\mathbb{R}^n$, we have that F is a function. Now we need to prove that F is an isometry, we will show that F is the composition of two isometries. We consider the linear mapping:

$$\begin{aligned} H : \mathbb{R}^n &\longrightarrow \mathbb{R}^n,\\ x &\mapsto F(x) - F(x_0).\end{aligned}$$

That is, for all $\lambda_i \in \mathbb{R}$,

$$\begin{aligned} H\left(\sum_{i=1}^{n} \lambda_i x_i\right) &= F\left(x_0 + \sum_{i=1}^{n} \lambda_i(x_i - x_0)\right) - F(x_0)\\ &= F(x_0) + \sum_{i=1}^{n} \lambda_i(F(x_i) - F(x_0)) - F(x_0)\\ &= \sum_{i=1}^{n} \lambda_i H(x_i).\end{aligned}$$

Hence H is a linear mapping. Now, we know $d(x_i, x_j) = d(y_i, y_j) = d(F(x_i), F(x_j))$ for all i, j. Then, for all i, j:

$$\langle x_i - x_j, x_i - x_j\rangle = \langle F(x_i) - F(x_j), F(x_i) - F(x_j)\rangle.$$

More over, for all i:

$$\begin{aligned} ||x_i|| &= \langle x_i,\, x_i\rangle \\ &= \langle x_i - x_0,\, x_i - x_0\rangle \\ &= \langle F(x_i) - F(x_0),\, F(x_i) - F(x_0)\rangle \\ &= \langle H(x_i),\, H(x_i)\rangle \\ &= ||H(x_i)||. \end{aligned}$$

Furthermore, for all i, j:

$$\begin{aligned} ||x_i - x_j|| &= ||F(x_i) - F(x_j)|| \\ &= ||F(x_i) - F(x_0) + F(x_0) - F(x_j)|| \\ &= ||[F(x_i) - F(x_0)] - [F(x_j) - F(x_0)]|| \\ &= ||H(x_i) - H(x_j)||. \end{aligned}$$

Since $||x_i - x_j||^2 = ||x_i||^2 - 2\langle x_i,\, x_j\rangle + ||x_j||^2$ and $||H(x_i) - H(x_j)||^2 = ||H(x_i)||^2 - 2\langle H(x_i),\, H(x_j)\rangle + ||H(x_j)||^2$ for all i, j, we obtain, for all i, j, that $\langle x_i,\, x_j\rangle = \langle H(x_i),\, H(x_j)\rangle$.

Finally, let x, $y \in \mathbb{R}^n$, we can write them as $x = \sum_{i=1}^n \lambda_i x_i$ and $y = \sum_{i=1}^n \gamma_i x_i$ where γ_i, $\lambda_i \in \mathbb{R}$ for all i. Therefore:

$$\begin{aligned} (d(x,\, y))^2 = ||x - y||^2 &= ||\sum_{i=1}^n \lambda_i x_i - \sum_{i=1}^n \gamma_i x_i||^2 \\ &= ||\sum_{i=1}^n (\lambda_i - \gamma_i) x_i||^2 \\ &= \langle \sum_{i=1}^n (\lambda_i - \gamma_i) x_i,\, \sum_{j=1}^n (\lambda_j - \gamma_j) x_j\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n (\lambda_i - \gamma_i)(\lambda_j - \gamma_j)\langle x_i,\, x_j\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n (\lambda_i - \gamma_i)(\lambda_j - \gamma_j)\langle H(x_i),\, H(x_j)\rangle \\ &= \langle \sum_{i=1}^n (\lambda_i - \gamma_i) H(x_i),\, \sum_{j=1}^n (\lambda_j - \gamma_j) H(x_j)\rangle \\ &= ||\sum_{i=1}^n (\lambda_i - \gamma_i) H(x_i)||^2 \\ &= ||\sum_{i=1}^n \lambda_i H(x_i) - \sum_{i=1}^n \gamma_i H(x_i)||^2 = (d(H(x),\, H(y)))^2. \end{aligned}$$

We deduce that H is an isometry. Then F is a composition of two isometries and therefore it is an isometry, more over, by definition $F(x_k) = y_k$ for all k. □

Theorem 2.3. *Let $X = \{x_0, \cdots, x_n\} \subset \mathbb{R}^n$ an affinely independent set. Let $\{y_0, \cdots, y_n\} \subset \mathbb{R}^n$ a set. If $d(x_i,\, x_j) = d(y_i,\, y_j)$ for all i, j, then there exists a unique isometry $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the property $F(x_k) = y_k$ for all k.*

Proof. (Existence). We consider the translation $t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x \mapsto x - x_0$. We have that $t(X) = \{0 = t(x_0), \cdots, t(x_n)\}$. By the previous lemma there exists an isometry $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

with the property $G(t(x_k)) = y_k$ for all k. Then there exist an isometry $F := (G \circ t) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the property $F(x_k) = y_k$ for all k.

(Uniqueness). We assume that F_1, $F_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are two isometries with the property $F_i(x_k) = y_k$, for all i, k. Since isometries preserve affine combinations and X is an affinely independent set, then $F_1 = F_2$. □

Observation 2.3. *Since every isometry $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ preserve affine combinations and are bijectives, by the previous theorem we conclude the following: If $\{x_0, \cdots, x_n\} \subset \mathbb{R}^n$ is an affinely independent set, $Y = \{y_0, \cdots, y_n\} \subset \mathbb{R}^n$ is a set and $d(x_i, x_j) = d(y_i, y_j)$ for all i, j, then Y is an affinely independent set.*

Theorem 2.4. *Let $X \subset \mathbb{R}^n$ a set. Let $f : X \longrightarrow X$ an isometry. If $\mathbb{R}^n$ is the affine span of X, then there exists a unique isometry $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the property $F|_X = f$.*

Proof. Let $\{x_0, \cdots, x_n\} \subset X$ an affinely independent set, since f is an isometry we have that $d(x_i, x_j) = d(f(x_i), f(x_j))$ for all i, j. By Theorem 2.3, there exists a unique isometry $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the property $F(x_k) = f(x_k)$ for all k. Furthermore, by Observation 2.3, $T := \{f(x_0), \cdots, f(x_n)\} = \{F(x_0), \cdots, F(x_n)\}$ is an affinely independent set.

Now, if $x \in X$, then $d(f(x), f(x_k)) = d(x, x_k) = d(F(x), F(x_k)) = d(F(x), f(x_k))$ for all k. Since T is an affinely independent set, by Theorem 2.2, $f(x) = F(x)$. That is, $F|_X = f$. □

Observation 2.4. *By the previous theorem we conclude: If $\mathbb{R}^n$ is the affine span of a subset A, then $f : A \longrightarrow A$ is a bijective isometry if and only if there exists an isometry $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the property $F|_A = f$ and $F(A) = A$, that is, F is a symmetry of A and $F|_A = f$.*

3 Convex sets

In this section we discuss properties of convex sets in $\mathbb{R}^n$ and their vertices. A vertex in a convex set A is a element of A that does not belong to the line segment connecting any two distinct element of A. We denote the set of all vertices of a convex set A by $extr(A)$.

Observation 3.1. *If $A \subset B \subset \mathbb{R}^n$, then $extr(B) \cap A \subset extr(A)$.*

Lemma 3.1. *Let x, y, $z \in \mathbb{R}^n$ different points and f a symmetry of $\mathbb{R}^n$. We have that $x \in (y, z)$ if and only if $f(x) \in (f(y), f(z))$. That is, $f(extr(A)) = extr(f(A))$.*

Proof. By definition, $x \in (y, z)$ if and only if $x = (1 - \lambda)y + \lambda z$. Since f is injective, it is equivalent to say $f(x) = f((1 - \lambda)y + \lambda z)$. By Proposition 2.1, it is equivalent to say $f(x) = (1 - \lambda)f(y) + \lambda f(z)$, that is, $f(x) \in (f(y), f(z))$. □

Let A be a non-empty set in $\mathbb{R}^n$. We denote by $\underline{Convex(A)}$ the convex hull of A.

From now the following theorem is used, in particular, in the proof of Proposition 3.1. The proof of the theorem is given in [4]: 'Teorema 1.1.1 (Caratheodory)'.

Theorem 3.1 (Caratheodory)**.** *Let A be a non-empty set in $\mathbb{R}^n$. If $x \in Convex(A)$, then $x = \sum_{i=1}^{n+1} \lambda_i x_i$, where $\lambda_i \in \mathbb{R}_0^+$, $x_i \in A$ and $\sum_{i=1}^{n+1} \lambda_i = 1$.*

Observation 3.2. *By Caratheodory's Theorem and Proposition 2.1 we conclude that: If A is a non-empty subset of $\mathbb{R}^n$ and if $f \in Sym(A)$, $x \in Convex(A)$ if and only if $f(x) \in Convex(f(A))$. That is, $f(Convex(A)) = Convex(f(A))$.*

Proposition 3.1. *Let A be a non-empty set in $\mathbb{R}^n$ whose elements are all the vertices of $Convex(A)$. Then f is a symmetry of A if and only if f is a symmetry of $Convex(A)$.*

Proof. ($\Longleftarrow$) If f is a symmetry of $Convex(A)$, then by Definition 2.1 $f(Convex(A)) = Convex(A)$, and therefore $f(A) \subset Convex(A)$. Since $A = extr(Convex(A))$, necessarily $f(A) = A$, because Lemma 3.1 says that f sends vertices to vertices, that is:

$$f(A) = f(extr(Convex(A))) = extr(f(Convex(A))) = extr(Convex(A)) = A$$

We conclude that f is a symmetry of A.

($\Longrightarrow$) Let $f \in Sym(A)$, that is $f(A) = A$, by last observation:

$$f(Convex(A)) = Convex(f(A)) = Convex(A)$$

We conclude that f is a symmetry of $Convex(A)$. □

Observation 3.3. *From last proof we conclude: If A is any non-empty subset of $\mathbb{R}$ and if $f \in Sym(A)$, then $f \in Sym(Convex(A))$.*

Lemma 3.2. *Let $x = \sum_{i=1}^{t} \lambda_i x_i \in \mathbb{R}^n$ with $t > 1$, $x_1, \cdots, x_t \in \mathbb{R}^n$ and $\lambda_1, \cdots, \lambda_t \in \mathbb{R}_0^+$ such that $\sum_{i=1}^{t} \lambda_i = 1$. If $\lambda_1 \neq 0 \neq \lambda_2$, $\gamma := \sum_{i=2}^{t} \lambda_i$ and if $y := \frac{1}{\gamma} \sum_{i=2}^{t} \lambda_i x_i$, then $x = y = x_1$ or $x \in (y,\, x_1)$ with $y \neq x_1$.*

Proof. Since $\gamma + \lambda_1 = 1$ and $\lambda_1 \neq 0 \neq \lambda_2$, then $\gamma,\, \lambda_1 \in (0,\, 1)$. Therefore we obtain:

$$\gamma y + \lambda_1 x_1 = \sum_{i=2}^{t} \lambda_i x_i + \lambda_1 x_1 = x.$$

Hence $x = y = x_1$ or $x \in (y,\, x_1)$ with $y \neq x_1$. □

Theorem 3.2. *Let A be a non-empty set in $\mathbb{R}^n$. The points of A are all the vertices of the convex hull of A if and only if A satisfies the following property: for all $x \in A$, x is not in the convex hull of $A \setminus \{x\}$.*

Proof. ($\Longrightarrow$) If $A = \{x\}$ it is immediate. Otherwise we assume there exists $x \in A$ such that $x \in Convex(A \setminus \{x\})$. Hence $x = \sum_{i=1}^{t} \lambda_i x_i$ with $x_i \in A \setminus \{x\}$ for all i, $x_i \neq x_j$ for $i \neq j$, $\lambda_i \in \mathbb{R}_0^+$ for all i and $\sum_{i=1}^{t} \lambda_i = 1$. Therefore $t > 1$ and at least two coefficients λ_l, λ_k with $l \neq k$ are non-zero, because $x \neq x_i$ for all i. Since they are arbitrarily indexed, without loss of generality we can say that $\lambda_1 \neq 0 \neq \lambda_2$. Since $x \in A = extr(Convex(A))$, by Lemma 3.2 we have $x = x_1$, generating a contradiction. We conclude that if $A = extr(Convex(A))$ then, for all $x \in A$, $x \notin Convex(A \setminus \{x\})$.

($\Longleftarrow$) We need to prove that $A = extr(Convex(A))$.

First we will prove that $extr(Convex(A)) \subset A$ because we won't use the hypothesis. If $A = Convex(A)$ then obviously $extr(Convex(A)) \subset A$. Otherwise we can consider $x \in Convex(A) \setminus A$, then we have $x = \sum_{i=1}^{t} \lambda_i x_i$ with $x_i \in A$ for all i, $x_i \neq x_j$ for $i \neq j$, $\lambda_i \in \mathbb{R}_0^+$ for all i and $\sum_{i=1}^{t} \lambda_i = 1$. Since $x \notin A$, at least two coefficients λ_l, λ_k with $l \neq k$ are non-zero. As before, by Lemma 3.2, we have that $x = x_1 \in A$ or exists $y \in Convex(A)$ such that $x \in (y,\, x_1)$. Since $x \neq x_1$, we obtain that $x \notin extr(Convex(A))$. This means, if $x \in extr(Convex(A))$, then $x \notin Convex(A) \setminus A$, that is, $x \in A$. We conclude that $extr(Convex(A)) \subset A$.

Now we will prove that $A \subset extr(Convex(A))$. We assume there exists $x \in A$ such that $x \in (y,\, z) \subset Convex(A)$. We have that $x = (1 - \lambda)y + \lambda z$ where $\lambda \in (0,\, 1)$, $y = \sum_{i=1}^{n+1} \delta_i y_i$ and $z = \sum_{i=1}^{n+1} \epsilon_i z_i$ with $y_i,\, z_i \in A$, $\delta_i,\, \epsilon_i \in \mathbb{R}_0^+$ for all i and $\sum_{i=1}^{n+1} \delta_i = 1 = \sum_{i=1}^{n+1} \epsilon_i$. Since $(1 - \lambda) \sum_{i=1}^{n+1} \delta_i + \lambda \sum_{i=1}^{n+1} \epsilon_i = 1$, we have $x \in Convex(\{y_i,\, z_i\}_{i=1}^{n+1}) \subset Convex(A)$, where $\{y_i,\, z_i\}_{i=1}^{n+1} \subset A$. By our hypothesis, $x \notin Convex(B)$ for all $B \subset A \setminus \{x\}$, because $Convex(B) \subset Convex(A \setminus \{x\})$, we conclude that $\{y_i,\, z_i\}_{i=1}^{n+1} \not\subset A \setminus \{x\}$ but $\{y_i,\, z_i\}_{i=1}^{n+1} \subset A$, that is, we need to $x = y_j$ or $x = z_j$ for some $j \geq 1$ with non-zero coefficient.

If all δ_l or if all ϵ_k with $y_l \neq x \neq z_k$ are zero, then $y = \sum \delta_j x = x \sum \delta_j = x$ or $z = \sum \epsilon_j x = x \sum \epsilon_j = x$. Since $y \neq x \neq z$, we have that some δ_l and some ϵ_k with $y_l \neq x \neq z_k$ are non-zero, then we can write $x = \gamma_0 x + \sum_{i=1}^{t} \gamma_i x_i$ where $\gamma_i \in \{(1-\lambda)\delta_i,\, \lambda\epsilon_i\}_{i=1}^{n+1} \subset \mathbb{R}_0^+$ for all $i > 0$, $\gamma_0 \in (0,\, 1)$, $\gamma_j \neq 0$ for some $j > 0$ and $\gamma_0 + \sum_{i=1}^{t} \gamma_i = 1$ such that $x_i \in A \setminus \{x\}$ for all i. Then $x = \frac{1}{1-\gamma_0} \sum_{i=1}^{t} \gamma_i x_i$ with $\frac{1}{1-\gamma_0} \sum_{i=1}^{t} \gamma_i = 1$, but this means that $x \in Convex(A \setminus \{x\})$, that is, we have a contradiction with the hypothesis. We conclude that $x \in extr(Convex(A))$, and therefore $A \subset extr(Convex(A))$. □

Observation 3.4. *Note that in fact, we proved that if $x \in A$ but $x \notin Convex(A \setminus \{x\})$, then $x \in extr(Convex(A))$.*

Observation 3.5. *From last prove we conclude that, for all non-empty set $A \subset \mathbb{R}^n$, we have that $extr(Convex(A)) \subset A$.*

Corollary 3.1. *Let A be a non-empty set in $\mathbb{R}^n$. If A satisfies the following property: for all $x \in A$ there exists a convex set H_x such that $x \notin H_x$ and $A \setminus \{x\} \subset H_x$, then the points of A are all the vertices of the convex hull of A.*

Proof. Let $x \in A$ and H_x a convex set such that $x \notin H_x$ and $A \setminus \{x\} \subset H_x$. Since H_x is convex, we have that $Convex(A \setminus \{x\}) \subset H_x = H_x \setminus \{x\}$.

Since x was any element of A, we conclude by last theorem. □

Observation 3.6. *Also, in particular, if $x \in A$ and if there exists a convex set H_x such that $x \notin H_x$ but $A \setminus \{x\} \subset H_x$, then $x \in extr(Convex(A))$.*

Corollary 3.2. *Let $A = \{x_1, \cdots, x_t\} \in \mathbb{R}^n$ be an affinely independent set. Then $A = extr(Convex(A))$.*

Proof. If $t = 1$ it is immediate.

Let $t > 1$. Since the elements of A are arbitrarily indexed, without loss of generality we assume $x_1 \in Convex(A \setminus \{x_1\})$. Then $x_1 = \sum_{i=2}^{t} \lambda_i x_i$ with $\lambda_i \geq 0$ for all i, at least two coefficients λ_l, λ_k with $l \neq k$ are non-zero and $\sum_{i=2}^{t} \lambda_i = 1$. This means, $0 = \sum_{i=2}^{t} \lambda_i (x_i - x_1)$, i.e., A is affinely dependent, so we have a contradiction with the hypothesis. Since x_1 really was any element of A, we conclude that for all $x \in A$ we have $x \notin Convex(A \setminus \{x\})$. By the previous theorem we obtain $A = extr(Convex(A))$. □

4 A regular $n-$simplex

In this section we prove the existence of a regular n-simplex in $\mathbb{R}^{n-1}$, i.e., we need the convex hull of an affinely independent set X of n-points such that $d(x,\, y) = r$ for all $x,\, y \in X$ with $x \neq y$ and for some real constant r. Furthermore, we need X to have a barycenter. We also show all necessary properties of a regular $n-$simplex below.

Definition 4.1. *Let us consider $S_n = (e_n)$, the canonical basis in $\mathbb{R}^n$. We define the standard n-simplex as the convex hull of S_n and denote it by Δ^n.*

Proposition 4.1. *The standard n-simplex in $\mathbb{R}^n$ is isometric to a regular $n-$simplex in $\mathbb{R}^{n-1}$.*

Proof. Immediately, we have that Δ^n is the convex hull of an affinely independent set of $n-$points. Furthermore, $d(x,\, y) = \sqrt{2}$ for all $x,\, y \in S_n$ with $x \neq y$.

Now consider $t = \frac{1}{n} \sum_{i=1}^{n} e_i \in \Delta^n$, we have $d(x,\, t) = \sqrt{(n-1) \cdot \frac{1}{n^2} + (1 - \frac{1}{n})^2}$ regardless of the chosen element $x \in S_n$. Then, t is a barycenter of S_n.

Since Δ^n is in the convex hull of S_n, then Δ^n is in the affine span of S_n, that is a hyperplane of

$\mathbb{R}^n$. Since our hyperplane is isometric to $\mathbb{R}^{n-1}$, we can identify S_n and Δ^n with some sets X_0 and Z_0 respectively in $\mathbb{R}^{n-1}$ with the same distance between its elements. We conclude that X_0 is an affinely independent set with $Z_0 = Convex(X_0)$, therefore Z_0 is a regular n-simplex in $\mathbb{R}^{n-1}$ and we verify its existence. □

Observation 4.1. *Let X_0 and Z_0 be the affinely independent set and the n-regular simplex respectively obtained in the preceding proof, in $\mathbb{R}^{n-1}$. With $Z_0 = Convex(X_0)$ and $\sqrt{2} = d(x, y)$ for all $x, y \in X_0$ with $x \neq y$.*

a) By Corollary 3.2 $X_0 = extr(Z_0)$. Now we can say that the elements of X_0 are the vertices of Z_0. Furthemore, if Y is the set of the vertices of a regular n-simplex W, then Y is affinely independent, because $W = Convex(V)$ where V is an affinely independent set and therefore $Y = extr(W) = extr(Convex(V)) = V$.

b) If f is a symmetry of $\mathbb{R}^{n-1}$, then $\sqrt{2} = d(x, y)$ for all $x, y \in f(X_0)$ with $x \neq y$, $f(X_0)$ is also an affinely independent set, by Proposition 2.1, and $f(Z_0)$ is also a regular n-simplex, by Observation 3.2. Furthermore, $f(X_0) = extr(f(Z_0))$ by Lemma 3.1.

c) We denote $X_0 = \{x_1, \cdots, x_n\}$. Let $f_1 : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$, $x \mapsto x - x_1$ be a symmetry of $\mathbb{R}^{n-1}$. Then we denote $Z_1 := f_1(Z_0)$ and $X_1 := f_1(X_0)$ as a regular n-simplex and the set of its vertices respectively. Also we still have to $\sqrt{2} = d(x, y)$ for all $x, y \in X_1$ with $x \neq y$, but now we know $0 \in X_1$. Furthermore, $\{x_2, \cdots, x_n\}$ is a basis of $\mathbb{R}^{n-1}$.

d) Let $f_\epsilon : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$, $x \mapsto \epsilon x$ be a biyective linear mapping, $\epsilon \in \mathbb{R}^+$. Then $Z_\epsilon := f_\epsilon(Z_1) = f_\epsilon(Convex(X_1)) = Convex(f(X_1))$ is a regular n-simplex with the set of all its vertices $X_\epsilon := f_\epsilon(X_1)$, $0 \in X_\epsilon$ and $\epsilon\sqrt{2} = d(x, y)$ for all $x, y \in f_\epsilon(X_1)$ with $x \neq y$. Furthermore, $X_\epsilon \setminus \{0\}$ is a basis of $\mathbb{R}^{n-1}$.

Proposition 4.2. *In Observation 4.1, let $Y = \{y_1, \cdots, y_n\}$ be the vertices of a regular n-simplex $W := Convex(Y)$ such that $d(x, y) = \epsilon\sqrt{2}$ for all $x, y \in Y$ with $x \neq y$, for some $\epsilon \in \mathbb{R}^+$. Then there exists $g \in Sym(\mathbb{R}^{n-1})$ such that $Y = g(X_\epsilon)$, and therefore $W = g(Z_\epsilon)$.*

Proof. Let $X_\epsilon = \{0 = x_1^{(\epsilon)}, \cdots, x_n^{(\epsilon)}\}$ where $d(x, y) = \epsilon\sqrt{2}$ for all $x, y \in X_\epsilon$ with $x \neq y$. We define $g : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$ by $g\left(x_1^{(\epsilon)} + \sum_{i=2}^n \lambda_i(x_i^{(\epsilon)} - x_1^{(\epsilon)})\right) = g(x_1^{(\epsilon)}) + \sum_{i=2}^n \lambda_i(g(x_i^{(\epsilon)}) - g(x_1^{(\epsilon)}))$, where $g(x_i^{(\epsilon)}) = y_i$ and $\lambda_i \in \mathbb{R}$ for all i. By the proof of Lemma 2.2, we have that g is an isometry. Since g is an isometry then $g \in Sym(\mathbb{R}^{n-1})$. By the definition of g we conclude that $g(X_\epsilon) = Y$, hence $W = Convex(Y) = Convex(g(X_\epsilon)) = g(Convex(X_\epsilon)) = g(Z_\epsilon)$. □

Observation 4.2. *By the previous proposition we conclude the following: If Y_1 and Y_2 are two regular n-simplex in $\mathbb{R}^{n-1}$ with arbitrarily indexed vertices $\{y_1^{(1)}, \cdots, y_n^{(1)}\}$ and $\{y_1^{(2)}, \cdots, y_n^{(2)}\}$ respectively, where the vertices have the property $d(y_i^{(1)}, y_j^{(1)}) = d(y_l^{(2)}, y_k^{(2)})$ for all i, j, l, k with $i \neq j$ and $l \neq k$. Then there exists an isometry g such that $g(y_i^{(1)}) = y_i^{(2)}$ for all i.*

Theorem 4.1. *Let $S = \{A_1, \cdots, A_n\}$ be the vertices of a regular n-simplex in $\mathbb{R}^{n-1}$ centered at t, that is, $t = \sum_{i=1}^n \frac{1}{n} A_i$. Let r be the distance between t and the elements of S. Then there is only one ball of radius r containing S, it is the ball centered at t.*

Proof. First, let us see the standard n-simplex in $\mathbb{R}^n$. Let H be the hyperplane affinely spanned by the standard basis $\{e_i\}_{i=1}^n$ in $\mathbb{R}^n$. We will prove that the only (n-1)-ball of radius $d(t_0, e_1)$ in H containing Δ^n is the ball centered at $t_0 := \sum_{i=1}^n \frac{1}{n} e_i$.
It is sufficient to demonstrate that t_0 is the only element of H with the property:

$$d(t_0, e_1) = Inf\{\delta_y \geq 0 \mid y \in H \wedge \delta_y = max_{1 \leq i \leq n}\{d(y, e_i)\}\}.$$

We define $f_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, $x = (x_1, \cdots, x_n) \mapsto (d(x, e_j))^2 = (1 - x_j)^2 + \sum_{i=1, i\neq j}^n x_i^2$; $g_0 : \mathbb{R}^n \longrightarrow \mathbb{R}$, $x = (x_1, \cdots, x_n) \mapsto \sum_{i=1}^n x_i$; $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, $x = (x_1, \cdots, x_n) \mapsto x_j$; all of them belonging to $\mathcal{C}^1$. Now let us find the minimum of f_j subject the constraint $g_0(x) = 1$ and $g_j(x) = c < \frac{1}{n}$ where c is a constant, this is possible because $\nabla g_0(x) \neq 0 \neq \nabla g_j(x)$ for all x. Now our constraint is the non-empty and unbounded intersection between two hyperplanes, that is, a non-empty convex set which is complete, and by the theorem of minimizing vector f_j has a unique minimum subject to the constraint, also f_j has not a maximum subject to the constraint because the set isn't bounded. Now our minimum x_c satisfies for some λ_1, $\lambda_2 \in \mathbb{R}$:

$$\begin{aligned} \nabla f_j(x_c) &= \lambda_1 \nabla g_0(x_c) + \lambda_2 \nabla g_j(x_c), \\ g_0(x_c) &= 1, \\ g_j(x_c) &= c\,. \end{aligned}$$

That is, with $x_c = (x_1, \cdots, x_n)$:

$$\begin{aligned} (2x_1, \cdots, -2(1 - x_j), \cdots, 2x_n) &= \lambda_1(1, \cdots, 1) + \lambda_2(0, \cdots, \underbrace{1}_{j}, \cdots 0), \\ \sum_{i=1}^n x_i &= 1, \\ x_j &= c < \frac{1}{n}\,. \end{aligned}$$

Then $x_c = (\frac{\lambda_1}{2}, \cdots, \underbrace{c}_{j}, \cdots \frac{\lambda_1}{2})$ with $c = 1 + \frac{\lambda_1 + \lambda_2}{2}$ and $c + \sum_{i=1, i\neq j}^n \frac{\lambda_1}{2} = 1$. Hence $c = 1 - \frac{\lambda_1}{2}(n-1)$ and therefore $\frac{\lambda_1}{2} = \frac{1-c}{n-1} > \frac{1}{n}$. Furthermore $f_j(x_c) = \frac{\lambda_1^2}{4}(n-1)^2 + \sum_{i=1, i\neq j}^n \frac{\lambda_1^2}{4} = n(n-1)\frac{\lambda_1^2}{4}$, since $\frac{\lambda_1}{2} > \frac{1}{n}$, we have that $f_j(x_c) > n(n-1)\frac{1}{n^2} = (n-1)^2\frac{1}{n^2} + (n-1)\frac{1}{n^2} = (1 - \frac{1}{n})^2 + \sum_{i=1, i\neq j}^n \frac{1}{n^2} = f_j(t_0)$ for all $c < \frac{1}{n}$.

Finally, if $x \in H \setminus \{t_0\}$, then $x = \sum_{i=1}^n \gamma_i e_i$, where $\sum_{i=1}^n \gamma_i = 1$, this means that $\gamma_j < \frac{1}{n}$ for some j. Since $g_0(x) = 1$ and $g_j(x) = \gamma_j < \frac{1}{n}$, then $f_j(x) \geq f_j(x_{\gamma_j}) > f_j(t_0)$, i.e., $d(x, e_j) > d(t_0, e_j)$.

We deduce that the only $(n-1)$-ball in H of radius $d(t_0, e_1)$ containing Δ^n is the ball centered at t_0. Then by Proposition 4.1, Observation 4.1 and Proposition 4.2 we conclude this theorem. □

Lemma 4.1. *Let x, y, $z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that $x = (1 - \lambda)z + \lambda y = z + \lambda(z - y)$.*

a) If $\lambda \in (0, 1)$, then $x \in (y, z)$.

b) If $\lambda \in \mathbb{R}^-$, then $z \in (x, y)$.

c) If $\lambda \in \mathbb{R}_{>1}$, then $y \in (x, z)$.

Proof. a) If $\lambda \in (0, 1)$, then by definition $x \in (y, z)$.

b) If $\lambda \in \mathbb{R}^-$, then $1 - \lambda > 1$ and $\frac{1}{1-\lambda} \in (0, 1)$. We have that $\frac{x}{1-\lambda} = z + \frac{\lambda y}{1-\lambda}$, then $\frac{1}{1-\lambda}x + \frac{-\lambda}{1-\lambda}y = z$, and since $\frac{1}{1-\lambda} + \frac{-\lambda}{1-\lambda} = 1$, we finally obtain $z \in (x, y)$.

c) If $\lambda \in \mathbb{R}_{>1}$, then $\frac{1}{\lambda} \in (0, 1)$. We have that $\frac{x}{\lambda} = \frac{(1-\lambda)z}{\lambda} + y$, then $\frac{1}{\lambda}x + \frac{-(1-\lambda)}{\lambda}z = y$, and since $\frac{1}{\lambda} + \frac{-(1-\lambda)}{\lambda} = 1$, we finally obtain $y \in (x, z)$.

□

Proposition 4.3. *Let $S = \{A_1, \cdots, A_n\}$ be the vertices of a regular n-simplex centered at the origin in $\mathbb{R}^{n-1}$, with $n \geq 3$. We consider $C_0 := \dfrac{A_1 + 2A_2 + \cdots + nA_n}{1 + 2 + \cdots + n}$, and thus the line $\overleftrightarrow{0C_0}$. Then $\overleftrightarrow{0C_0} \cap S = \emptyset$.*

Proof. We denote $\sigma := 1 + \cdots + n$. If $\overleftrightarrow{0C_0} \cap S \neq \emptyset$, then $A_j = \lambda 0 + (1-\lambda)C_0$ for some j and for some $\lambda \in \mathbb{R} \setminus \{0, 1\}$. Since $Convex(S)$ is centered at the origin, then $0 = \sum_{k=1}^{n} \frac{1}{n} A_k$. Thus we have:

$$\begin{aligned} A_j &= \lambda 0 + (1-\lambda)C_0 \\ &= \lambda \sum_{k=1}^{n} \left(\frac{1}{n}\right) A_k + (1-\lambda) \sum_{k=1}^{n} \left(\frac{k}{\sigma}\right) A_k \\ &= \sum_{k=1}^{n} \left(\frac{\lambda}{n} + \frac{(1-\lambda)k}{\sigma}\right) A_k \ . \end{aligned}$$

Therefore we have for all $k \neq j$:

$$0 = \frac{\lambda}{n} + \frac{(1-\lambda)k}{\sigma}.$$

In particular, for $p \equiv j - 1 \, mod \, n$ and $q \equiv j + 1 \, mod \, n$ with $p, q \in [1, n]$, we have that:

$$\frac{\lambda}{n} + \frac{(1-\lambda)p}{\sigma} = \frac{\lambda}{n} + \frac{(1-\lambda)q}{\sigma}.$$

That is, $p = q$, this implies that $j - 1 \equiv j + 1 \, mod \, n$, we have a contradiction because $n \geq 3$. Since we had only assumed that $\overleftrightarrow{0C_0} \cap S \neq \emptyset$, we conclude that $\overleftrightarrow{0C_0} \cap S = \emptyset$. □

5 Hyperplanes

In this section we see all necessary results on real hyperplanes. In $\mathbb{R}^n$ a hyperplane is any translation of a vector subespace whose dimension is $n - 1$. If H is a hyperplane, it has the form $H = H[y, r] := \{x \in \mathbb{R}^n \mid \langle x, y \rangle = r\}$, where r and y are fixed elements of $\mathbb{R}$ and $\mathbb{R}^n$ respectively. We also denote the two semi-spaces determined by $H[y, r]$ as $H^-[y, r] := \{x \in \mathbb{R}^n \mid \langle x, y \rangle < r\}$ and $H^+[y, r] := \{x \in \mathbb{R}^n \mid \langle x, y \rangle > r\}$.

We use the notation $B[t, r]$ for a closed ball of radius r centered at t and $B(t, r)$ for the respective open ball.

Proposition 5.1. *For any $y \in \mathbb{R}^n$ and any $r \in \mathbb{R}$ we have that $H[y, r]$, $H^-[y, r]$, $H^+[y, r]$, $H^-[y, r] \cup H[y, r]$ and $H^+[y, r] \cup H[y, r]$ are convex.*

Proof. Let $\lambda \in (0, 1)$. If $x, z \in \mathbb{R}^n$ such that $\langle x, y \rangle ? r$ and $\langle z, y \rangle ? r$, where $? \in \{=, <, >, \leq, \geq\} =: !$. If $w = (1-\lambda)x + \lambda z \in (x, z)$, then:

$$\langle w, y \rangle = \langle (1-\lambda)x + \lambda z, y \rangle = (1-\lambda)\langle x, y \rangle + \lambda \langle z, y \rangle ? (1-\lambda)r + \lambda r = r \ .$$

We obtain that w is in the same set as x and z. Since $?$ was any element of $!$, x, z were any elements of their set and w was any element of the line segment connecting x and z; we conclude as requested. □

Observation 5.1. *By Corollary 3.1, if a non-empty set $A \in \mathbb{R}^n$ satisfies the property: for all $x \in A$ there exists $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$ such that $x \in H[y, r]$ and $A \setminus \{x\} \subset H^-[y, r]$ or $A \setminus \{x\} \subset H^+[y, r]$, then the points of A are all the vertices of the convex hull of A. In particular, if for $x \in A$ there exists $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$ such that $x \in H[y, r]$ and $A \setminus \{x\} \subset H^-[y, r]$ or $A \setminus \{x\} \subset H^+[y, r]$, then $x \in extr(Convex(A))$.*

Lemma 5.1. *Let $r \in \mathbb{R}^+$. If $v \in \partial B[0,\, r] \subset \mathbb{R}^n$, we know $-v \in \partial B[0,\, r]$, but also:*

a) $0 \in (-v,\, v)$.

b) For all $y \in B[0,\, r] \setminus \{v\}$, $\langle v,\, v\rangle > \langle y,\, v\rangle$.

c) For all $y \in B[0,\, r] \setminus \{-v\}$, $\langle y,\, v\rangle > -\langle v,\, v\rangle$.

Proof. a) It is sufficient to note that $0 = \frac{1}{2}v - \frac{1}{2}v = \frac{1}{2}v + \frac{1}{2}(-v)$.

b) Let $y \in B[0,\, r] \setminus \{v\}$, then:

$$\begin{aligned} 0 &< \langle y - v,\, y - v\rangle \\ &= \langle y,\, y\rangle - 2\langle y,\, v\rangle + \langle v,\, v\rangle \\ &\leq r^2 - 2\langle y,\, v\rangle + r^2 \\ &= -2\langle y,\, v\rangle + 2\langle v,\, v\rangle. \end{aligned}$$

That is, $\langle v,\, v\rangle > \langle y,\, v\rangle$.

c) Let $y \in B[0,\, r] \setminus \{-v\}$, then:

$$\begin{aligned} 0 &< \langle y + v,\, y + v\rangle \\ &= \langle y,\, y\rangle + 2\langle y,\, v\rangle + \langle v,\, v\rangle \\ &\leq r^2 + 2\langle y,\, v\rangle + r^2 \\ &= +2\langle y,\, v\rangle + 2\langle v,\, v\rangle. \end{aligned}$$

That is, $-\langle v,\, v\rangle < \langle y,\, v\rangle$.

□

Lemma 5.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x \mapsto rx$ a linear mapping, where $r \in \mathbb{R} \setminus \{0\}$. Then for all $a,\, b,\, c,\, d \in \mathbb{R}^n$ we have: $\langle a,\, b\rangle < \langle c,\, d\rangle$ if and only if $\langle f(a),\, f(b)\rangle < \langle f(c),\, f(d)\rangle$.*

Proof. Let $a,\, b,\, c,\, d \in \mathbb{R}^n$.

We first assume that $\langle a,\, b\rangle < \langle c,\, d\rangle$. Then:

$$\begin{aligned} \langle f(a),\, f(b)\rangle &= \langle ra,\, rb\rangle \\ &= r^2\langle a,\, b\rangle \\ &< r^2\langle c,\, d\rangle \\ &= \langle f(c),\, f(d)\rangle. \end{aligned}$$

Now we assume that $\langle f(a),\, f(b)\rangle < \langle f(c),\, f(d)\rangle$. Then:

$$\begin{aligned} \langle a,\, b\rangle &= \frac{1}{r^2}\langle ra,\, rb\rangle \\ &= \frac{1}{r^2}\langle f(a),\, f(b)\rangle \\ &< \frac{1}{r^2}\langle f(c),\, f(d)\rangle \\ &= \frac{1}{r^2}\langle rc,\, rd\rangle \\ &= \langle c,\, d\rangle. \end{aligned}$$

□

Proposition 5.2. *Let $r \in \mathbb{R}^+$ and $x, y \in \partial B[0, r] \subset \mathbb{R}^n$. We denote $M := \{z \in \mathbb{R}^n \mid d(z, x) \geq d(y, x)\}$. Then $M \cap \partial B[0, r] \subset H^-[x, \langle y, x\rangle] \cup H[x, \langle y, x\rangle]$.*

Proof. Let $z \in M \cap \partial B[0, r]$. We have that $\langle x, x\rangle = r^2 = \langle z, z\rangle = \langle y, y\rangle$, then:

$$\begin{aligned} d(z, x) &= \langle z - x, z - x\rangle = 2r^2 - 2\langle z, x\rangle, \\ d(y, x) &= \langle y - x, y - x\rangle = 2r^2 - 2\langle y, x\rangle. \end{aligned}$$

Since $z \in M$, then $d(z, x) \geq d(y, x)$, and by the last equations we conclude that $\langle z, x\rangle \leq \langle y, x\rangle$. Hence $z \in H^-[x, \langle y, x\rangle] \cup H[x, \langle y, x\rangle]$. Since z was any element of our set, we finally obtain that $M \cap \partial B[0, r] \subset H^-[x, \langle y, x\rangle] \cup H[x, \langle y, x\rangle]$. □

Proposition 5.3. *Let $a, c \in \mathbb{R}^n$. If $b \in H^-[a, \langle c, a\rangle]$ (respectively $b \in H^+[a, \langle c, a\rangle]$), then $H^-[a, \langle b, a\rangle] \cup H[a, \langle b, a\rangle] \subsetneq H^-[a, \langle c, a\rangle]$ (respectively we have $H^+[a, \langle b, a\rangle] \cup H[a, \langle b, a\rangle] \subsetneq H^+[a, \langle c, a\rangle]$).*

Proof. If $b \in H^-[a, \langle c, a\rangle]$, then $\langle b, a\rangle < \langle c, a\rangle$, and therefore:

$$\begin{aligned} H^-[a, \langle b, a\rangle] \cup H[a, \langle b, a\rangle] = \{x \in \mathbb{R}^n \mid \langle x, a\rangle \leq \langle b, a\rangle\} &\subsetneq \{x \in \mathbb{R}^n \mid \langle x, a\rangle < \langle c, a\rangle\} \\ &= H^-[a, \langle c, a\rangle]. \end{aligned}$$

The other case is similar. □

Lemma 5.3. *Let $q_1, q_2, p \in \mathbb{R}$ and $v \in \mathbb{R}^n$ be such that $q_1 < q_2$. Then $pv \in (q_1 v, q_2 v)$ if and only if $p \in (q_1, q_2)$.*

Proof. We have the following equivalence chain:

$$\begin{aligned} pv \in (q_1 v, q_2 v) &\iff pv = q_1 v + \lambda(q_2 v - q_1 v) \text{ for some } \lambda \in (0, 1) \\ &\iff pv = (q_1 + \lambda(q_2 - q_1))v \text{ for some } \lambda \in (0, 1) \\ &\iff p = q_1 + \lambda(q_2 - q_1) \text{ for some } \lambda \in (0, 1) \\ &\iff p \in (q_1, q_2). \end{aligned}$$

□

Theorem 5.1. *Let $r \in \mathbb{R}^+$ and $s \in \mathbb{N}$. Let $v \in \partial B[0, r] \subset \mathbb{R}^n$ be fixed. Now let $\{C_1, \cdots, C_s\} \subset \partial B[0, r] \setminus \{v\}$. Then there exists $j \in [-1, 1)$ such that $\{C_1, \cdots, C_s\} \subset H^-[v, \langle kv, v\rangle]$, for all real number $k > j$.*

Proof. We consider a fixed element, $C_l \in \{C_1, \cdots, C_s\}$, such that $d(C_l, v) = min\{d(C_i, v) \mid i \in [1, s]\}$. By Proposition 5.2, we immediately have:

$$\{C_1, \cdots, C_s\} \subset H^-[v, \langle C_l, v\rangle] \cup H[v, \langle C_l, v\rangle]. \tag{5}$$

Let $j := \frac{\langle C_l, v\rangle}{\langle v, v\rangle}$, we will prove that this j is the j we are looking for. Since $v \neq 0$, then $\langle v, v\rangle > 0$, and therefore $j \in \mathbb{R}$. By Lemma 5.1 we obtain that $-\langle v, v\rangle \leq \langle C_l, v\rangle < \langle v, v\rangle$, that is, $j \in [-1, 1)$. Furthermore, $jv \in H[v, \langle C_l, v\rangle]$ because $\langle jv, v\rangle = \langle C_l, v\rangle$. Then, with equation 5, we have:

$$\{C_1, \cdots, C_s\} \subset H^-[v, \langle jv, v\rangle] \cup H[v, \langle jv, v\rangle]. \tag{6}$$

Now let $k > j$. Hence $k\langle v, v\rangle > j\langle v, v\rangle$, that is, $\langle kv, v\rangle > \langle jv, v\rangle$. Thus we obtain $jv \in H^-[v, \langle kv, v\rangle]$, and by Proposition 5.3 we have $H^-[v, \langle jv, v\rangle] \cup H[v, \langle jv, v\rangle] \subset H^-[v, \langle kv, v\rangle]$. Then, with equation 6, we obtain:

$$\{C_1, \cdots, C_s\} \subset H^-[v, \langle kv, v\rangle]. \tag{7}$$

Since k was any element greater than j, we conclude that j has the required properties.

□

Observation 5.2. *In Theorem 5.1, if $k \neq 0$ and if $x \in \mathbb{R}^n$, then $\langle x, v\rangle = \langle kv, v\rangle$ if and only if $\langle x, kv\rangle = \langle kv, kv\rangle$, that is, $H[v, \langle kv, v\rangle] = H[kv, \langle kv, kv\rangle]$, and therefore we obtain that $\{C_1, \cdots, C_s\} \subset H^-[kv, \langle kv, kv\rangle]$.*

6 Balls

In this section we cover results on balls that are basic but essential for our goal.

Remark 6.1. *In $\mathbb{R}^n$, we will show that $Convex(\partial B[0,\, r]) = B[0,\, r]$, $\partial B[0,\, r] = extr(B[0,\, r])$.*

a) For any $r \in \mathbb{R}^+$, the ball $B(0,\, r)$ is convex.

Proof. Let $x,\, y \in B(0,\, r)$ and $z \in (x,\, y)$. Then $z = (1-\lambda)x + \lambda y$ with $\lambda \in (0,\, 1)$. Hence $||z|| = ||(1-\lambda)x + \lambda y|| \leq ||(1-\lambda)x|| + ||\lambda y|| = (1-\lambda)||x|| + \lambda||y|| < (1-\lambda)r + \lambda r = r$, that is, $z \in B(0,\, r)$. Since they were arbitrary elements, we conclude that the open ball is convex. □

b) Since for any $r \in \mathbb{R}^+$ we know that $B[0,\, r]$ is the closure of $B(0,\, r)$ and $B(0,\, r)$ is convex, then $B[0,\, r]$ is convex.

c) Let $w \in B(0,\, r)$. By Lemma 5.1 $0 \in Convex(\partial B[0,\, r])$. Now we assumed that $w \neq 0$. Thus we have that $w \in \left(0,\, \dfrac{rw}{||w||}\right)$ because $\dfrac{||w||}{r} \in (0,\, 1)$ and $w = \left(1 - \dfrac{||w||}{r}\right)0 + \dfrac{||w||}{r}\left(\dfrac{rw}{||w||}\right)$. Since $\dfrac{rw}{||w||} \in \partial B[0,\, r]$, then $w \in Convex(\partial B[0,\, r])$. That is, $B(0,\, r) \subset Convex(\partial B[0,\, r])$, then $B[0,\, r] = \partial B[0,\, r] \cup B(0,\, r) \subset Convex(\partial B[0,\, r])$. Finally, since $B[0,\, r]$ is convex and $\partial B[0,\, r] \subset B[0,\, r]$, then $B[0,\, r] = Convex(\partial B[0,\, r])$.

d) For all $v \in \partial B[0,\, r]$ we know $v \in H[v,\, \langle v,\, v\rangle]$. Then by Lemma 5.1 we can say that $\partial B[0,\, r] \setminus \{v\} \subset H^-[v,\, \langle v,\, v\rangle]$. Then by Observation 5.1 v is a vertex of the convex hull of $\partial B[0,\, r]$. We conclude that $\partial B[0,\, r] = extr(Convex(\partial B[0,\, r])) = extr(B[0,\, r])$.

e) Since all the translations are symmetries of $\mathbb{R}^n$, by Observation 2.2, $B(v,\, r)$ and $B[v,\, r]$ are convex for all $v \in \mathbb{R}^n$ and $r \in \mathbb{R}$. Furthermore, by Observation 3.2 and Lemma 3.1, $Convex(\partial B[v,\, r]) = B[v,\, r]$ and $\partial B[v,\, r] = extr(B[v,\, r])$.

Remark 6.2. *Let $S = \{A_1,\, \cdots,\, A_n\}$ be the vertices of a regular n-simplex in $\mathbb{R}^{n-1}$. Let $t_0 := \sum_{i=1}^{n} \frac{1}{n} A_i$ the center of the regular n-simplex and $r := d(A_j,\, t_0)$ for all j. Since $S \subset \partial B[t_0,\, r]$ and $B[t_0,\, r]$ is convex, then the regular n-simplex is a subset of $B[t_0,\, r]$.*

7 Finding the polytope

In this section we see and detail the proof of Marcos Cossarini (2021).

Theorem 7.1. *If G is a finite group, then there exists a convex polytope M whose symmetry group is G and $\mathbb{R}^{n-1}$ is the affine span of M.*

Proof. Let n be the order of G. We will find our polytope in $\mathbb{R}^{n-1}$. Since we know the cases for $n = 1$ and $n = 2$, we assume that $n \geq 3$.

First, thanks to Proposition 4.1 and a good translation, let us consider a regular n-simplex in $\mathbb{R}^{n-1}$ centered at the origin with vertices $S = \{A_1,\, A_2,\, ...,\, A_n\}$, that is, $0 = \frac{1}{n}\sum_{i=1}^{n} A_i \in Convex(S)$.

By Section 2: if f is a simmetry of S, then $f(0) = 0$, therefore f is a $\mathbb{R}-$vector space morphism.
Since S is an affine basis of $\mathbb{R}^{n-1}$, then f is determined by the images of the elements in S. By Observation 4.2 we conclude that $Sym(S) \cong S_n$. In this way we can identify G as a subgroup

of $Sym(S)$, we will continue talking about G but now as a fixed subgroup of $Sym(S)$. We remark that $Sym(S)$ and therefore G are groups of linear isometries, in particular preserve the norm and the inner product.

Now use the notation $C_0 := \dfrac{A_1 + 2A_2 + \cdots + nA_n}{1 + 2 + \cdots + n} \in Convex(S)$. Since $C_0 \neq 0$, we can consider $C := \dfrac{rC_0}{||C_0||} \in \partial B[0, r]$ where $r = d(0, A_i) = ||A_i||$ for all i. By Proposition 4.3 we obtain that $S \subset \partial B[0, r] \setminus \{C\}$ because C is in the line $\overleftrightarrow{0C_0}$. By Theorem 5.1 we obtain a particular $j \in [-1, 1)$, and we consider $k = max\left\{\dfrac{1}{2}, \dfrac{1+j}{2}\right\} \in (max\{0, j\}, 1)$ with the property that $S \subset H^-[C, \langle kC, C\rangle]$. Furthermore, since $k \in (0, 1)$, then $||kC|| = k||C|| < ||C|| = r$, and therefore $kC \in B(0, r)$.

We denote $P := G \cdot kC$ and $T = S \cup P$. We note that $P \subset B(0, r)$ because $G \cdot kC \subset G \cdot B(0, r) = B(0, r)$, whence $T \subset B[0, r]$. Furthermore, $T \cap \partial B[0, r] = S$ and $S \cap P = \emptyset$.

We will prove that $Sym(T) = G$. Since $f(S) = S$ and $f(P) = P$ for all $f \in G$, then $f(T) = T$ for all $f \in G$, that is, $G \leq Sym(T)$. Now, if $f \in Sym(T)$, then $S \subset T = f(T) \subset f(B[0, r]) = B[b, r]$, where $b \in \mathbb{R}^n$ is such that $f \equiv g + b$ with g a linear map, but by Theorem 4.1 we necessarily have that $b = 0$, so that f is a linear map. Since f is an isometry and a linear map, then $f(\partial B[0, r]) = \partial B[0, r]$. Therefore $f(S) = f(T \cap \partial B[0, r]) = f(T) \cap f(\partial B[0, r]) = T \cap \partial B[0, r] = S$, so that $f \in Sym(S)$. Since $f(S) = S$, then $f(S^c) = S^c$, and thus $f(P) = f(T \cap S^c) = f(T) \cap f(S^c) = T \cap S^c = P$, that is, $f \in Sym(P)$. Since $f \in Sym(P)$, by definition of P, there exists $g \in G$ such that $f(kC) = g(kC)$. Since g is a linear map and $g \in Sym(S)$, then:

$$\begin{aligned} g\left(\frac{A_1 + 2A_2 + \cdots + nA_n}{1 + 2 + \cdots + n}\right) &= \frac{g(A_1) + 2g(A_2) + \cdots + ng(A_n)}{1 + 2 + \cdots + n} \\ &= \frac{A_{g(1)} + 2A_{g(2)} + \cdots + nA_{g(n)}}{1 + 2 + \cdots + n}. \end{aligned}$$

Since f is a linear map and $f \in Sym(S)$, then:

$$\begin{aligned} f\left(\frac{A_1 + 2A_2 + \cdots + nA_n}{1 + 2 + \cdots + n}\right) &= \frac{f(A_1) + 2f(A_2) + \cdots + nf(A_n)}{1 + 2 + \cdots + n} \\ &= \frac{A_{f(1)} + 2A_{f(2)} + \cdots + nA_{f(n)}}{1 + 2 + \cdots + n}. \end{aligned}$$

Thus we obtain that $A_{f(i)} = A_{g(i)}$ for all i because $f(kC) = g(kC)$. Since S is an affine basis of $\mathbb{R}^{n-1}$, we conclude that $f = g$, that is, $f \in G$. Then $Sym(T) \leq G$, and we conclude that $Sym(T) = G$.

Finally we will prove that $T = extr(Convex(T))$ to conclude that $Sym(Convex(T)) = G$. By Observation 3.5 we know that $extr(Convex(T)) \subset T$. Since $T \subset B[0, r]$ and $B[0, t]$ is convex, then $Convex(T) \subset B[0, r]$. Therefore we obtain that $S = T \cap \partial B[0, r] = T \cap extr(B[0, r]) \subset Convex(T) \cap extr(B[0, r]) \subset extr(Convex(T))$. Furthermore, by the choice of our kC we have $S \subset H^-[C, \langle kC, C\rangle]$. Since $k \neq 0$, by Observation 5.2, we obtain $S \subset H^-[kC, \langle kC, kC\rangle]$. Since G is a group of linear isometries, then $P = G \cdot kC \subset \partial B[0, ||kC||]$, and therefore by Lemma 5.1 we have $P \setminus \{kC\} \subset H^-[kC, \langle kC, kC\rangle]$. Thus we obtain $T \setminus \{kC\} = S \cup P \setminus \{kC\} \subset H^-[kC, \langle kC, kC\rangle]$, since $kC \in H[kC, \langle kC, kC\rangle]$, by Observation 5.1 we have that $kC \in extr(Convex(T))$. Since $G = Sym(T)$, then by Observation 3.3 $G \leq Sym(Convex(T))$, and therefore by Lemma 3.1 $P = G \cdot kC \subset G \cdot extr(Convex(T)) = extr(G \cdot Convex(T)) = extr(Convex(T))$, because $kC \in extr(Convex(T))$. Since $S, P \subset extr(Convex(T))$, then $T \subset extr(Convex(T))$. Thus its follows that $T = extr(Convex(T))$.

We deduce by Proposition 3.1 that $Sym(Convex(T)) = Sym(T) = G$. Since T is a finite set, then we conclude that $M := Convex(T)$ is a convex polytope whose symmetry group is G. Finally, since $S \subset M$, then $\mathbb{R}^{n-1}$ is the affine span of M. □

8 Conclusion

In the *Unidad de Investigación* it was necessary to learn more about affine spaces and convex sets to perform the proofs of this report. After using almost all of the proofs in this report to go through each step of the final proof, there is no doubt that Marcos Cossarini's (2021) proof is correct and the answer to the initial question is affirmative: For every finite group G we can construct a convex polytope M, in some Euclidean space, whose symmetry group is G.

If one wanted to achieve something analogous with symmetries, in a Euclidean space, for any groups, due to the cardinality of the groups, one would have to work with a larger collection of sets than just with the convex polytopes. The recommendation is first to specify the next question for finitely generated groups, one step higher than finite groups.

Another question to be clarified could be: What does it tell us that two sets have the same group of symmetries? With this report as a starting point, we invite more people to investigate group symmetries and what they can tell us about groups.

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